# Tori and their Destruction

- integrability ⇒ can write as action-angle form:

$$\begin{cases} \frac{d\underline{I}}{dt} = 0, \quad \frac{d\underline{\theta}}{dt} = \underline{\omega}(\underline{I}) \\ \text{const } \underline{I}. \end{cases}$$

⇒ motion defines tori

$$\frac{d\theta}{dt} = \omega_1(I_1)\, t$$

$$\frac{d\phi}{dt} = \omega_2(I_2)\, t$$

scanning $I_1, I_2$ (linked to $E$)
⇒ define nested tori

etc.

$$\left\{ \begin{array}{l} \text{e.g. box} \\ \omega_1 = \pi^2 I_1 / m a^2 \\ \omega_2 = \pi^2 I_2 / m b^2 \\ \\ E = I_1\omega_1 + I_2\omega_2 \end{array} \right.$$

- motion on each toroidal surface will cover surface egodically, unless $\frac{\omega_1}{\omega_2}$ rational.
- many surfaces ⇒ define volume of phase space.

– motion is conditionally periodic

i.e. ergodic motion on bounded surface
⇒ Poincare recurrence guarantees nearby return to i.c.

⇒ How robust are toroidal surfaces?

i.e. if $H \to H_0(\underline{I}) + \epsilon H_1(\underline{I}, \underline{\theta})$

symmetry breaking perturbation

Can we integrate the perturbed system to some order in $\epsilon$?

i.e. transform $\underline{I}, \underline{\theta} \to \underline{J}, \underline{\phi}$

$$\text{s/t} \quad \left.\begin{aligned} \dot{\underline{J}} &= 0 \\ \dot{\underline{\phi}} &= \omega(\underline{J}) \end{aligned}\right\} \text{ to specified order in P.T.?}$$

This is equivalent to exploring fragility of "surfaces" ⇒ i.e. can nested structure be maintained with $O(\epsilon)$ deformation?

n.b. ⇒ intro to canonical perturbation theory

→ start with $\underline{\underline{1}}$ deg freedom:

$$J = I + O(\epsilon)$$
$$\phi = \theta + O(\epsilon)$$

then: old: $I, \theta$

new: $J, \phi$

s/t $\dot{J} = 0$ to $O(\epsilon)$

so have C.T. problem:

$p \Leftrightarrow I$
$q \Leftrightarrow \theta$
(old)

$P = J$
$Q = \phi$
(new)

so

indep
$q \Leftrightarrow \theta$
$P \Leftrightarrow J$

dep
$p \Leftrightarrow I$
$Q = \phi$

$$p = \frac{\partial F}{\partial q} = \frac{\partial S}{\partial q}$$

$F = S$ here
$S = $ H-J fctn.

so

$$I = \partial S/\partial \theta$$

$$\phi = \partial S/\partial J$$

where: $S = S_0 + \epsilon \underbrace{S_1}_{\text{unknown}}$

$$= J\theta + \epsilon S_1$$

now here:

$$S = S_0 + \epsilon S_1$$

$H'(J) \equiv K(J)$ → re-label.

new, integrated Hamiltonian → fctn of $J$, only

and can expand:

$$K(J) = K_0(J) + \epsilon K_1(J) + \ldots\ldots$$

$\circledast$

$$K(J) = H(I, \theta)$$

$$= H_0\left(\frac{\partial S}{\partial \theta}, \theta\right) + \epsilon H_1\left(\frac{\partial S}{\partial \theta}, \theta\right) + \ldots\ldots$$

n.b.: $S' = S_0 + \epsilon S_1$
$= J\theta + \epsilon S_1$

$$I = J + \epsilon \frac{\partial S_1}{\partial \theta} \Rightarrow J = I - \epsilon \frac{\partial S_1}{\partial \theta}$$

$$\phi = \theta + \epsilon \frac{\partial S_1}{\partial J} \qquad \phi = \theta + \epsilon \frac{\partial S_1}{\partial J}$$

now, plugging $S$ into relation for $H' = K$, etc

$$K_0(J) + \epsilon K_1(J) + \epsilon^2 K_2(J)$$

$$= H_0\left(J + \epsilon \frac{\partial S_1}{\partial \theta} + \epsilon^3 \frac{\partial S_2}{\partial \theta} + \cdots\right)$$

$$+ \epsilon H_1\left(J_1 + \epsilon \frac{\partial S_1}{\partial \theta} + \ldots, \theta\right)$$

cranking expansion to $O(\epsilon^2)$:

$$K_0(J) + \epsilon K_1(J) + \epsilon^2 K_2(J) + \cdots =$$

$$H_0(J) + \epsilon \frac{\partial S_1}{\partial \theta} \frac{\partial H_0}{\partial J} + \epsilon^2 \frac{\partial S_2}{\partial \theta} \frac{\partial H_0}{\partial J}$$

$$+ \epsilon H_1(J,\theta) + \epsilon^2 \frac{\partial S_1}{\partial \theta} \frac{\partial H_1(J)}{\partial J}$$

$$+ \tfrac{1}{2} \epsilon^2 \left(\frac{\partial S_1}{\partial \theta}\right)^2 \frac{\partial^2 H_0}{\partial J^2}$$

matching order-by-order:

$$H_0 = K_0$$

$$K_1(J) = \frac{\partial S_1}{\partial \theta} \frac{\partial H_0}{\partial J} + H_1(J,\theta)$$

$$K_2(J) = \frac{1}{2}\left(\frac{\partial S_1}{\partial \theta}\right)^2 \frac{\partial^2 H_0}{\partial J^2} + \frac{\partial S_2}{\partial \theta} \frac{\partial H_0}{\partial J}$$

$$+ \frac{\partial S_1}{\partial \theta} \frac{\partial H_1}{\partial J} + H_2$$

if present.

etc.

For $O(\epsilon)$:

$$K_1(J) = \frac{\partial S_1}{\partial \theta}\frac{\partial H_0}{\partial J} + H_1(J,\theta)$$

$$= \frac{\partial S_1}{\partial \theta}\underbrace{\omega_0(J)}_{\text{winding frequency}} + H_1(J,\theta)$$

where understand:

$$I = J + O(\epsilon)$$
$$\phi = \theta + O(\epsilon)$$

$$\boxed{\begin{aligned} \theta &= \phi - \epsilon\, \partial S_1/\partial J \\ I &= J + \epsilon \frac{\partial S_1}{\partial \theta} \end{aligned}}$$

Now, if define:

$$H_1 = \underbrace{\langle H_1 \rangle}_{\text{avg.}} + \underbrace{\tilde{H}_1}_{\theta \text{ dep piece (symmetry breaking)}}$$

$$\langle H_1 \rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} H_1$$

(mean part)

then

averaging $K_1(J)$ eqn $\Rightarrow$

$$\boxed{K_1(J) = \langle H_1 \rangle}$$

and for $S_1$, from solvability:

$$\omega_0(J)\frac{\partial S_1}{\partial \theta} = K_1(J) - H_1$$

$$= \cancel{K_1(J)} - \cancel{\langle H_1 \rangle} - \tilde{H}_1$$

$$= -\tilde{H}_1$$

$$\boxed{\omega_0(J)\frac{\partial S_1}{\partial \theta} = -\tilde{H}_1}$$

Now, from before, as motion closed and periodic:

$$\tilde{H}_1 = \sum_{n=1}^{\infty} H_n(J)\, e^{in\theta}$$

$$S_1 = \sum_{n=1}^{\infty} S_n\, e^{in\theta} \qquad S = J\theta + \epsilon S_1$$

$\Rightarrow$

$$\boxed{S_1 = -\sum_n \frac{H_n(J)}{in\omega_0(J)} e^{in\theta}}$$

so can finally write full solution to $O(\epsilon)$:

$$\phi = \theta + \epsilon \frac{\partial S_1}{\partial J}(J,\theta)$$

$$J = I - \epsilon \frac{\partial S_1}{\partial \theta}(J,\theta)$$

$$\omega = \omega_0(J) + \epsilon \frac{\partial}{\partial J} K_1(J)$$

where:

$$K_1 = \langle H_1 \rangle$$

$$S_1 = \sum_n \frac{i H_n(J)}{n\omega_0(J)} e^{in\theta}$$

so, in 1 d-o-f, can define strategy of perturbative 'integration'.

BUT, if # d-o-f's > 1:

$\theta \to \underline{\theta}$ (i.e. $\theta, \phi$ toroidal angles)

$n\omega_0(J) \to \underline{n} \cdot \underline{\omega}_0(\underline{J})$

(i.e. $\underline{n} \cdot \underline{\omega}_0 = n\omega_1(J_1) + m\omega_2(J_2)$
where $E = J_1\omega_1 + J_2\omega_2$)

then if

$\underline{n} \cdot \underline{\omega}_0(\underline{J}) \to 0$, denominator vanishes and perturbation theory fails

$\Rightarrow$ welcome to the "problem of small divisors"

$\Rightarrow$ identifies resonant surfaces

i.e. special surfaces of nested torus

where pitch of perturbation
$n/m$ = pitch of winding $\frac{\omega_2}{\omega_1}$.

These seem (and are) most fragile surfaces.

These surfaces are "resonant surfaces"

Classic example:

- tokamak field lines $\qquad m = n\,q(r)$

$$q(r) = m/n$$

$q(r)$ ↓ pitch of lines (note shear) $\qquad m/n$ ↓ pitch of perturbation

- wave particle $\qquad \mathbf{v} = \omega/k$

$\mathbf{v}$ ↓ particle velocity $\qquad \omega/k$ ↓ wave phase velocity

n.b. here time makes resonance

$\partial S/\partial t = H - H'$

① ⇒ in vicinity of resonant surfaces, perturbative integration fails!

② since rationals are set of measure $\mu \to 0$ on whole #'s, resonant surfaces are in some sense 'special'.

→ sneak preview

distortions called "islands" form

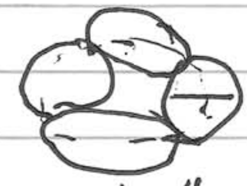
(const. H surface)

$I = 2$

→ + resonant perturbation

$m = 4$

$n = 2$



filamentation occurs.

(const. H. surface)

$w_I \sim \sqrt{\delta B}$

upshot: – trajectory undertakes excursion from surface but remains near

– phase space structure resembles that of pendulum.

→ caveat: secular canonical perturbation theory works for 1 resonance only!

strategy:

– remove resonance by transformation to frame co-rotating with resonant variables

Akin removal by frame change.

n.b. really avg. over fast variable

- limitation to removal of $\underline{\underline{1}}$ fast variable
  i.e. works as resonance $\Leftrightarrow$ slow

Now,

$$H = H_0(\underline{I}) + \epsilon H_1(\underline{I}, \underline{\theta})$$

if resonance: $\quad r\omega_1 - s\omega_2 \simeq 0$
$\Rightarrow$ resonance

$$\omega_1 = \frac{d\theta_1}{dt} \qquad \Rightarrow \qquad \theta = r\theta_1 - s\theta_2 \text{ is "slow".}$$

$$\omega_2 = \frac{d\theta_2}{dt}$$

so

$$(\underline{\omega}\cdot\nabla_{\underline{\theta}})\, f(\theta) = (\omega_1 \partial_{\theta_1} + \omega_2 \partial_{\theta_2}) f$$

$$= (r\omega_1 - s\omega_2) f_{r,s}$$

$\rightsquigarrow 0$, near resonance.

f dependence on $\theta$ is h.o. $\to$ slow.

thus, before:

$$\underline{I}, \underline{\theta} \rightarrow \underline{J}, \underline{\phi}$$

now:

$$\left.\begin{array}{l} I_1, \theta_1 \\ I_2, \theta_2 \end{array}\right\} \rightarrow \left\{\begin{array}{l} \overset{\text{slow}}{r\theta_1 - s\theta_2}, \quad \hat{J}_1 \\ \theta_2, \quad \hat{J}_2 \end{array}\right.$$

2 fast → 1 slow, 1 fast

$$F = S(\text{old positions, new momenta})$$

$$= S(\theta_1, \theta_2; \hat{J}_1, \hat{J}_2)$$

and type 2, so:

$$\boxed{S = \underbrace{(r\theta_1 - s\theta_2)\hat{J}_1 + \theta_2 \hat{J}_2}_{S_0} + \epsilon S_1}$$

so

$$I_1 = \partial S/\partial \theta_1 = r\hat{J}_1 + \epsilon\, \partial S_1/\partial \theta_1$$

$$I_2 = \partial S/\partial \theta_2 = (\hat{J}_2 - s\hat{J}_1) + \epsilon\, \partial S_1/\partial \theta_2$$

$$\phi_1 = \partial S/\partial \hat{J}_1 = r\theta_1 - s\theta_2 + \epsilon\, \partial S_1/\partial J_1$$

$$\phi_2 = \partial S/\partial \hat{J}_2 = \theta_2 + \epsilon\, \partial S_1/\partial J_2$$

$$H = H_0(\underline{I}) + \epsilon H_1(\underline{I}, \underline{\theta})$$

$$H_1 = \sum_{\ell, m} H_{\ell, m}(\underline{I})\, e^{i(\ell\theta_1 + m\theta_2)} \qquad \ell, m \neq 0$$

but know:

$$\phi_1 = r\theta_1 - s\theta_2 + \cancel{O(\epsilon)} \qquad \underline{\text{slow}}$$

$$\phi_2 = \theta_2 + \cancel{O}(\epsilon) \qquad \underline{\text{Fast}}$$

$\Rightarrow$

$$\theta_1 \cong (\phi_1 + s\phi_2)/r$$

$$\theta_2 \cong \phi_2$$

re-writing:

$$H_1 = \sum_{\ell, m} H_{\ell, m}(\underline{\hat{J}}) \exp\left[i\left(\frac{\ell}{r}\phi_1 + \frac{(\ell s + m r)}{r}\phi_2\right)\right]$$

$\left.\begin{array}{l}\phi_2 \to \text{fast} \\ \phi_1 \to \text{slow}\end{array}\right\}$ distinction only possible near resonance where $r\omega_1 - s\omega_2 \to 0$

[Now, average out fast $\phi_2$ dependence, and focus on evolution near resonance. $\Rightarrow$ isolates region near resonance

Thus, will have

$$K_1 = K_1(\underline{\hat{J}}, \underset{\substack{\uparrow \\ \text{slow}}}{\phi_1}) = \langle H_1 \rangle_{\phi_2}$$

$$\langle H_1 \rangle_{\phi_2} = \left\langle \sum_{\ell, m} H_{1\,\ell, m}(\underline{\hat{J}}) \exp\left[i\left(\frac{\ell}{r}\phi_1 + \frac{i}{r}(\ell s + m r)\phi_2\right]\right\rangle_{\phi_2}$$

or,

Simply put:

$$\frac{\ell}{m} = \frac{-r}{s}$$ $\Rightarrow$ mode # pitch of perturbation must match pitch of resonance

so

$$\sum_{\ell, m} \to \sum_{p(-r/s)}$$ $\Rightarrow$ sum over all harmonics of perturbation resonant

$\therefore$

$$\sum_{\ell, m} \to \sum_{p} F_{pr, ps}$$

upon $\phi_2$ integration: $\ell s = -mr$

$$\frac{\ell}{m} = \frac{-r}{s} \quad \text{but} \quad r\omega_1 - s\omega_2 \sim 0$$

$$\sim \frac{\omega_2}{\omega_1} \quad \Rightarrow \frac{\ell}{m} \text{ ratio set by resonance.}$$

so $H_{1\,\ell,m} \rightarrow H_{1\,-m\frac{r}{s},m}$ $\qquad \ell = \frac{-r}{s} m$

$\rightarrow H_{1\,-mr,\,ms}$

relabel

$\rightarrow H_{1\,-pr,\,ps}$

also $\frac{\ell}{r} = -\frac{m}{s}$ $\qquad$ relabel: $-\frac{m}{s} \rightarrow -m$

$-m \rightarrow -p$

so $\langle\ \rangle_{\phi_2}$ perturbation is

just harmonics of resonant pair $-r, s$.

$$\langle H_1 \rangle_{\phi_2} = \sum_{p=0}^{\infty} H_{1\,-rp,sp}\, e^{-ip\phi_1}$$

$\Rightarrow$

$$\boxed{\langle H \rangle = H_0(J) + \epsilon \sum_{p=0}^{\infty} H^{(1)}_{-pr,ps}\, e^{-ip\phi_1}}$$

From C-T rules:

$$\frac{\partial \langle H \rangle}{\partial \phi_2} = 0 \Rightarrow \frac{d}{dt}\hat{J}_2 = 0 \rightarrow$$ adiabatic invariant

and from C-T rules:

$$I_1 = r\hat{J}_1$$

$$I_2 = \hat{J}_2 - sJ_1$$

$\Rightarrow$ $\hat{J}_2 = I_2 + \frac{s}{r} I_1$ is adiabatic inv. of avgd Hamiltonian

$\phi_1$, $\phi_1$ ⟶ res.

so $\frac{d\vec{J}_2}{dt} = 0 \Rightarrow \frac{d\phi_2}{dt} = \frac{\partial\langle H\rangle}{\partial\vec{J}_2} \equiv \omega(\vec{J}_2)$

Now, $\langle H\rangle = \langle H(\vec{J}_1, \phi_1, \vec{J}_2)\rangle$

→ For solution, need understand motion on $\vec{J}_1$, $\phi_1$

→ without loss of generality, simplify by:

$p = 0, \pm 1$ harmonics only contribute

so

$$\langle H\rangle = H_0(\underline{\vec{J}}) + \epsilon H_{0,0}(\underline{\vec{J}})$$
$$+ 2\epsilon H_{r,-s}(\vec{J}) \cos\phi_1$$

$H_{-r,s} = H_{r,-s}$

and seek motion near fixed points, as characterization

so, $\dot{\hat{J}}_1 = 0$, $\dot{\phi}_1 = 0$ $\Rightarrow$ f.p. $\Leftrightarrow$ $\partial\langle H\rangle/\partial\phi_1 = 0$, $\partial\langle H\rangle/\partial\hat{J}_1 = 0$

there define: $\left.\begin{array}{l}\hat{J}_{1,0} = 0 \\ \phi_{1,0} = 0\end{array}\right\}$ fixed pts of motion

so

$$\frac{\partial \langle H \rangle}{\partial \phi} = 0 \Rightarrow -2\epsilon H^{(1)}_{r,-s} \sin\phi_1 = 0$$

$$\phi_1 = \underbrace{0}_{\text{fixed pts.}}\ \pm \pi$$

and

$$\frac{\partial \langle H \rangle}{\partial \hat{J}_1} = 0 \Rightarrow \frac{\partial H_0(\hat{J}_1)}{\partial \hat{J}_1} + \epsilon \frac{\partial H_{0,0}(\hat{J})}{\partial \hat{J}_1}$$

$$+ 2\epsilon \frac{\partial H^{(1)}_{r,-s}}{\partial \hat{J}_1} \cos\phi_1 = 0$$

Now

$$\frac{\partial}{\partial \hat{J}_1} = \frac{d I_1}{d \hat{J}_1} \frac{\partial}{\partial I_1} + \frac{\partial I_2}{\partial \hat{J}_1} \frac{\partial}{\partial I_2}$$

C-T rules $= r\frac{\partial}{\partial I_1} - s\frac{\partial}{\partial I_2}$

so, $\partial\langle H\rangle/\partial\hat{J}_1 = 0 \Rightarrow$ re-express

$$0 = \left(r\frac{\partial}{\partial I_1} - s\frac{\partial}{\partial I_2}\right) H_0(\underline{I})$$

$$+ \epsilon \frac{\partial}{\partial \hat{J}_1} H_{0,0} + 2\epsilon \frac{\partial H_{r,-s}^{(1)}}{\partial \hat{J}_1}\cos\phi_1$$

$$= \underbrace{(r\omega_1 - s\omega_2)}_{0 \text{ on resonance!}} + \epsilon\left(\frac{\partial H_{0,0}^{(1)}}{\partial \hat{J}_1} + 2\frac{\partial H_{r,-s}^{(1)}}{\partial \hat{J}_1}\cos\phi_1\right)$$

so, to lowest order:

$$\partial\langle H\rangle/\partial\hat{J}_1 = 0 \leftrightarrow d\phi_1/dt = 0$$

is satisfied by resonance condition.

so $\hat{J}_{1,0}$ defined by resonance condition.

so fixed points:

$\hat{J}_{1,0} \Leftrightarrow$ resonant position
$r\,\omega_1(\underline{J}) - s\,\omega_2(\underline{J}) = 0$

$\phi_{1,0} \Leftrightarrow \sin\phi_1 = 0$

n.b.
see 22a

so $\langle H \rangle = \langle H(\hat{J}_1, \hat{J}_2, \phi_1)\rangle$

$$= \langle H(\underbrace{\hat{J}_{1,0}}_{\text{resonance}} + \underbrace{\delta\hat{J}_1}_{\text{excursion}}, \phi_1 \mid \underbrace{\hat{J}_2}_{\text{IOM}})\rangle$$

so, expanding:

$$\langle H(\hat{J}_1, \phi_1)\rangle \cong H_0(\hat{J}_{1,0}) + \epsilon\left(H^{(1)}_{0,0}(\hat{J}_{1,0})\right)$$

$$+ \underbrace{\cancel{\frac{\partial H_0}{\partial \hat{J}_1}\bigg|_{\hat{J}_{1,0}}}}_{\text{reson.}} (\hat{J}_1 - J_{1,0}) + \frac{1}{2}\frac{\partial^2 H_0}{\partial \hat{J}_1^2}(\hat{J}_1 - J_{1,0})^2$$

$$+ 2\epsilon H^{(1)}_{r,-s}\cos\phi_1$$

$\Rightarrow$

$$\langle H(\hat{J}_1, \phi_1)\rangle \cong \text{const.} + \frac{1}{2}\frac{\partial^2 H_0}{\partial \hat{J}_1^2}(\hat{J}_1 - J_{1,0})^2 + 2\epsilon H^{(1)}_{r,-s}\cos\phi_1$$

so, have arrived at averaged Hamiltonian near resonance:

$$\boxed{\begin{aligned}\langle H(\hat{J}_1,\phi_1)\rangle &= \frac{1}{2}(\hat{J}_1-\hat{J}_{1,0})^2 \left.\frac{\partial^2 H_0}{\partial \vec{J}_1^2}\right|_{\hat{J}_{1,0}} - F\cos\phi_1 \\ &\equiv \frac{G}{2}(\hat{J}_1-\hat{J}_{1,0})^2 - F\cos\phi_1 \\ G = \left.\frac{\partial^2 H_0}{\partial \vec{J}_1^2}\right|_{\hat{J}_{1,0}}, &\quad F = 2\epsilon H^{(1)}_{r,s}\end{aligned}}$$

→ isomorphic to pendulum!

Recall for pendulum:

$$L = \frac{m}{2}\ell^2\dot{\theta}^2 - mg\ell(1-\cos\theta)$$

$$H = p\dot{\theta} - L = \frac{p_\theta^2}{2m\ell^2} - mg\ell\cos\theta$$

$$\Rightarrow \quad H(\hat{J}_1, \phi_1) = \frac{G}{2}(J_1 - J_{1,0})^2 - F\cos\theta$$

is form of Hamiltonian near resonance

Note:

- assumes $\dfrac{\partial^2 H_0}{\partial \hat{J}_1^2} = \dfrac{\partial \omega}{\partial \hat{J}_1} \neq 0$ (NL/shear)

"accidental" resonance.

- for properties:

$$\langle H(\hat{J}_1, \phi) \rangle = \frac{G}{2}(\bar{J}_1 - \bar{J}_{1,0})^2 - F\cos\phi_1$$

$\frac{G}{2}$: shear/NL parameter

$F$: perturbation amplitude

and so:

$$\Delta \dot{J} = -F\sin\phi_1$$
$$\dot{\phi}_1 = G\,\Delta J$$

$\phi_1 = 0 + \delta\phi$

$\Delta \ddot{J} + FG\,\Delta J = 0$

near $\phi_1 = 0$

i.e.

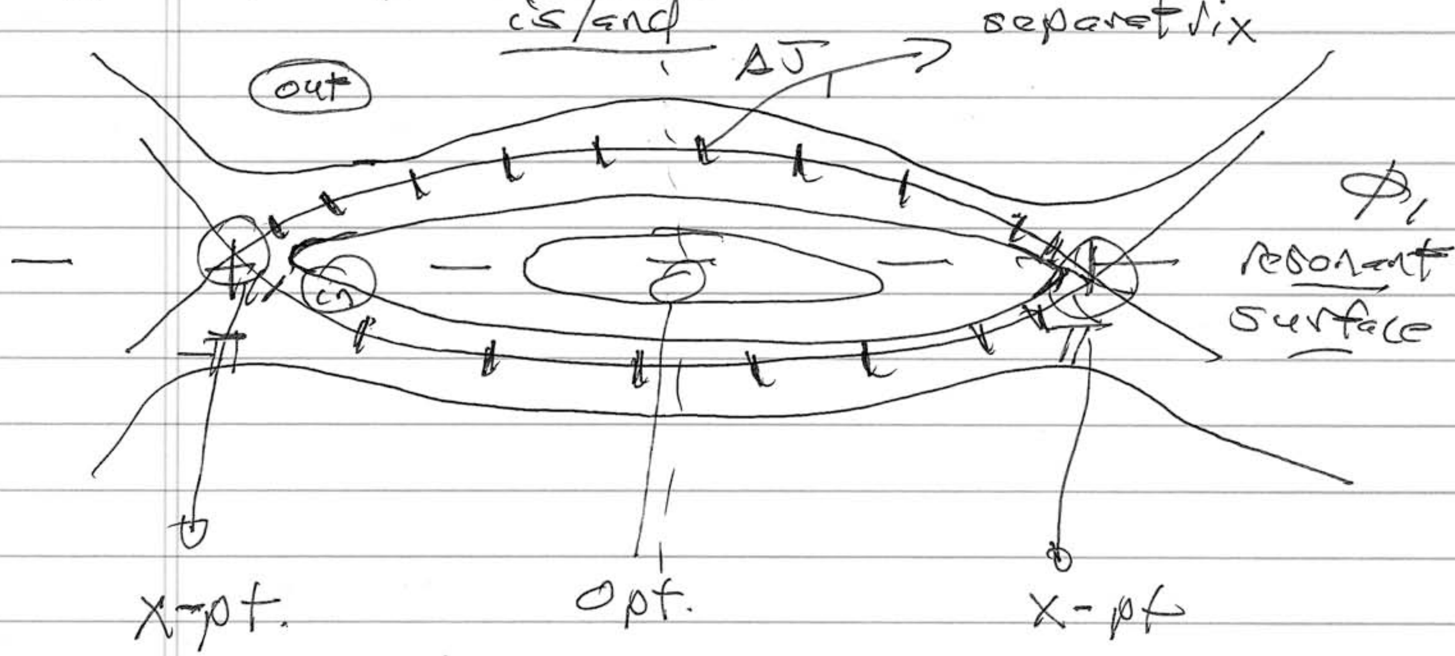
$$\Delta \ddot{J}_1 = -F\cos\phi_{1,0}\, G\, \Delta J$$

$$\Delta \ddot{J}_1 + FG\cos\phi_{1,0}\, \Delta J = 0$$

$FG > 0 \Rightarrow \phi_1 = 0$, stable fixed point
(o-pt/elliptic point)

$\phi_1 = \pm\pi$ $\Rightarrow$ unstable fixed pt.
(X-pt/hyperbolic pt.)

Contours: phase space island



→ stable fixed pt. ↔ elliptic point ↔ O pt.
- island center

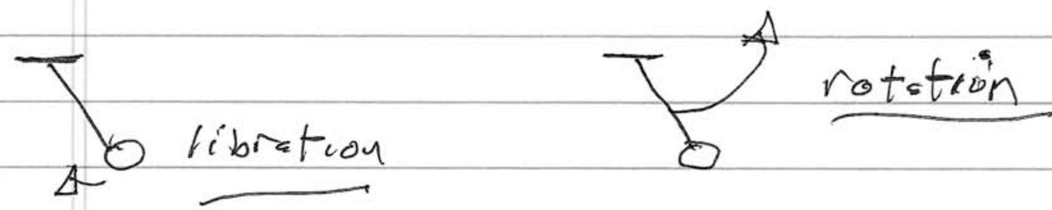
- center of trapped or libration region

→ unstable fixed point ↔ hyperbolic point ↔ X pt.

- island edge

- separatrix crossing point

→ separatrix 'separates' region of rotation (i.e. untrapped) from region of trapped (i.e. libration)




→ libration: elliptic orbits
rotation: hyperbolic orbits

- width of separatrix – "island width"

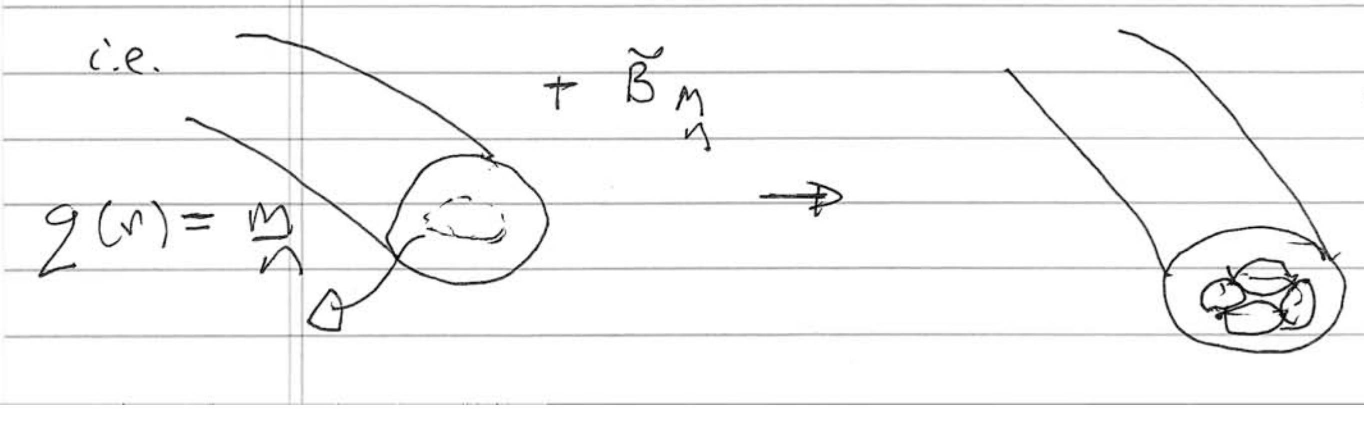
$$(\Delta J)_{max} \cong 2(F/G)^{1/2}$$
$$\cong 2\left(-2G H^{(1)}_{\ell_0 - s} \Big/ \left|\frac{\partial^2 H_0}{\partial J_1^2}\right|_{\tilde{J}_{1,0}}\right)^{1/2}$$

i.e. particle + wave:

$$H = (p + m\omega/k)^2/2m + q\phi_0 \cos kx$$
$$\Delta p = (q\phi_0 m)^{1/2}$$
$$\Delta v \cong (q\phi_0/m)^{1/2} \rightarrow \text{trapping width}$$

$\Rightarrow$ the Big Picture:

- resonant perturbations distort and foliate resonant tori in phase space, forming island chain structures.

i.e.



island chain on resonant surface.

Note:

- structure localized to resonant surface

- trapped / untrapped } orbits stay { trapped / untrapped.

- resonant surface is foliated but not destroyed.

- motion remains on surface, though surface is ruffled...